ON SEMI-INFINITE SYSTEMS OF LINEAR INEQUALITIES*

BY

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ABSTRACT

The ordered field $R(M)$ consists of the reals R with a transcendental M adjoined, which is larger than any real $r \in R$. Given any semi-infinite matrix (s.i.m.) interpreted as linear inequalities: $u^tP_i \geqq c_i$, $\forall_i \in I$, an arbitrary index set, it is also shown that the following are equivalent. (1) For every finite $J \subseteq I$ the system $u^tP_i \geqq c_i, i \in J$ is consistent, and (2) the s.i.m. has a solution $u \in R(M)^n$. Some consequences for "duality gaps" are also given.

The main result of this paper is Theorem 1. The ordered field $R(M)$ referred to in Theorem 1 and Lemma 1 consists of the reals R with a transcendental M adjoined and with ordering derived from the infinite valuation in which $a < M$ for all $a \in R$.

Theorem 1 is a compactness-type result, but, insofar as we know, it cannot be derived from the Compactness Theorem of Mathematical Logic which can insure the existence of a solution to (1) in *some* non-Archimedean elementary extension of R. $R(M)$ is not such an elementary extension since $\sqrt{M} \notin R(M)$.

The existence of solutions to (1) in $R(M)$ under the hypotheses of Theorem 1 is equivalent to the fact that the limiting behavior of "approximate solutions" to (1) in an appropriate locally convex topology can be replaced by algebraic characterizations parametrized by polynomials over $R(M)$.

These results were obtained by working from the duality theory of Ben-Israel, Charnes, and Kortanek (see [1]–[4]) and Kortanek [10], but are here proven using convexity theory directly. The results were announced earlier in [9].

1. The main result

Let an infinite set of linear inequalities be given and symbolized thus:

$$(1) \qquad u^tP_i \geqq c_i, \quad i \in I$$

* These results were obtained as part of the activities of the Management Science Research Group and School of Urban and Public Affairs, Carnegie-Mellon University.

Received October 25, 1970 and in revised form May 12, 1971

where I is an arbitrary set, $P_i \in R^n$ for all $i \in I$, and $c_i \in R$ for all $i \in I$. A vector $u \in F^n$ for which (1) holds, where F is an ordered field extending R, is called a *solution* to (1).*

LEMMA 1. *Suppose that, for every finite set index $J \subseteq I$, the finite system*

$$u^t P_i \geqq c_i, \qquad i \in J \tag{2}$$

has a solution in R^n. Let W be the space spanned by $\{P_i \mid i \in I\}$. Then either

(i) *The problem* (1) *has a solution in R^n, or*

(ii) *There exists a vector $u \in W$, $u \neq 0$, such that*

$$u^t \cdot P_i \geqq 0 \text{ for all } i \in I.$$

PROOF. Let $A = \{(P_i, -c_i) \mid i \in I\} \cup \{(0,1)\}$ where $(0,1) \in R^n \times R \cong R^{n+1}$. Let $C(A)$ be all non-negative linear combinations of the elements of A. Note that $C(A)$ is a subset of $W \times R$ and is in fact a convex cone.

Case 1. $C(A) = W \times R$.

Then there exist scalars $\lambda_i \geqq 0$, $i \in J$, where $J \subseteq I$ is finite, and a scalar $\mu \geqq 0$ such that

$$\sum_{i \in J} \lambda_i (P_i, -c_i) + \mu(0,1) = (0, -1).$$

where $(0, -1) \in W \times R$. By the hypotheses concerning (2), there exists an $x \in R^n$ satisfying (2), and so, by the above equation separated into its component equations,

$$\begin{aligned} 0 = 0 \cdot x = \sum_{i \in J} \lambda_i P_i x &\geqq \sum_{i \in J} \lambda_i c_i \\ &\geqq \sum_{i \in J} \lambda_i c_i - \mu = +1, \end{aligned}$$

a contradiction. Evidently, then, this case cannot arise.

Case 2. $C(A) \neq W \times R$

Since $C(A)$ is a cone, the origin must be a boundary point to $C(A)$, and hence, by the separating hyperplane theorem, there exists $v = (u, \alpha) \neq 0$, where $u \in W$, $\alpha \in R$, such that :

$$(u, \alpha) \cdot (P_i, -c_i) \geqq 0, \qquad i \in I$$

$$(u, \alpha) \cdot (0,1) \geqq 0.$$

* We call (1) a *semi*-infinite system of linear inequalities because n is finite.

The last equation gives $\alpha \geqq 0$. If $\alpha > 0$, from the first equation we get

$$u^t P_i \geqq \alpha c_i, \quad i \in I$$

i.e.,

$$\frac{u^t}{\alpha} P_i \geqq c_i, \quad i \in I.$$

Then (1) has a solution in R^n. If $\alpha = 0$, then the first equation gives

$$u^t P_i \geqq 0, \quad i \in I.$$

Since $v \neq 0$ and $\alpha = 0$, we have $u \neq 0$. Q.E.D.

THEOREM 1. *With the hypotheses and notation of Lemma* 1, *there is a solution u to* (1) *with $u \in R(M)^n$ such that all components of u are polynomials of degree not exceeding the dimension of W.*

PROOF. The proof is by induction on $d =$ the dimension of W.

For $d = 0$, evidently $P_i = 0$ for all $i \in I$, and so by the hypothesis regarding (1), each $c_i \leqq 0$, $i \in I$, and hence (1) has a solution in R^n and we obtain a polynomial solution in $R(M)^n$ of degree $\leqq d = 0$.

If $d > 0$, and (1) has a solution in R^n, we are again done. If (1) does not have a solution in R^n, then by Lemma 1 there exists a vector u such that $u_u^t \cdot P_i \geqq 0$, $i \in I$. We then define

$$Q = \{P_i \mid u^t P_i > 0\}$$

$$Q' = \{P_i \mid u^t P_i = 0\}.$$

Since $u \in W$ and $u \neq 0$, Q' has dimension $\leqq (d-1)$. Since the system

$$u^t P_i \geqq c_i, \ P_i \in Q'$$

satisfies hypotheses like (1), by induction we have the existence of a solution $\tilde{u} \in R(M)^n$ whose coordinates are polynomials of degree $\leqq d - 1$ such that

$$\tilde{u}^t P_i \geqq c_i, \quad \text{all } P_i \in Q'.$$

Then defining $\hat{u} = M^d u + \tilde{u}$ we see that $i \in Q' \to \hat{u}^t P_i = M^d u^t P_i + \tilde{u}^t P_i = 0 + \tilde{u}^t P_i = \tilde{u}^t P_i \geqq c_i$ and also $i \in Q - Q' \to \hat{u}^t P_i > c_i$ since $u^t P_i > 0$ and M^d exceeds $\tilde{u}$ in size by virtue of the larger exponent d. Thus we have

$$\hat{u}^t P_i \geqq c_i, \quad i \in I,$$

with all components of $\tilde{u}$ of degree $\leqq d$, and we are done. Q.E.D.

REMARK. The following example shows that the bound d on the maximal degree of a solution $u \in R(M)^n$ is best possible. Let $N =$ the set of positive integers. Every solution $u \in R(M)^n$ to the following inequalities has polynomial part of degree at least n:

$$\begin{array}{ll} u_1 \geqq k & \text{all } k \in N \\ u_2 \geqq ku_1 & \text{all } k \in N \\ \vdots & \vdots \\ u_n \geqq ku_{n-1} & \text{all } k \in N \end{array}$$

2. Some consequences of the main result

To investigate the situation when (1) has a "solution in the limit", we define a *weak solution* to (1) to be a sequence $u_1, u_2, \cdots$ in R^n such that, for each $i \in I$,

$$\lim_k \inf u_k^t P_i \geqq c_i \tag{3}$$

where $+\infty$ is allowed as a limit.

THEOREM 2. *The following are equivalent:*

(1) *There is a weak solution to* (2) *for every finite subset* $J \subseteq I$.

(2) *There is a solution to* (2) *for every finite subset* $J \subseteq I$.

(3) *There is a solution* $u \in R(M)^n$ *to* (1) *all of whose components are polynomials of degree* $\leqq n$.

(4) *There is a solution* $u \in R(M)^n$ *to* (1).

(5) *There is a weak solution to* (1).

PROOF.

(1) → (2). The is an easy application of Farkas' Lemma.

(2) → (3). Theorem 1.

(3) → (4). Trivial.

(4) → (5). Let $u = u(M) \in R(M)^n$ be a solution to (1). Set $u_k = u(k)$ for every integer $k \geqq 0$. It is then trivial to check that (3) holds.

(5) → (1). Trivial. Q.E.D.

REMARKS. (1) The equivalence of (2) and (5) in Theorem 2 is trivial for I countable, but we know of no proof for general I that does not essentially use Theorem 1. (2) In [8] an apparently more general notion of limiting convergence is discussed which employs arbitrary nets, and this notion is shown to be equivalent to weak convergence. Those nterested in the duality theory of Ben-Israel, Charnes, and Kortanek may wish to consult Theorem 6 of [8] for this stronger result. (3) The fact that (4) → (3) in Theorem 2 can be made constructive by stating a procedure which transforms any solution $u \in R(M)^n$ to a solution $\tilde{u} \in R(M)^n$ of degree $\leqq n$. The interested reader may consult Theorem 8 of [8].

We now give a result which explicates the nature of the duality gaps that have been observed in semi-infinite programming (see [4]).

First we shall need some terminology.

Let $P \in R^n$. We consider the semi-infinite linear program (I) and its dual (II):

(I)	(II)
$\sup \sum_{i \in I} \lambda_i c_i$	$\inf u^t P$
subject to $\sum_{i \in I} \lambda_i P_i = P$	subject to $u^t P_i \geqq c_i,\ i \in I$
$\lambda_i \geqq 0,\ i \in I$	

where the sums in (I) are to be such that $\{i \mid \lambda_i \neq 0\}$ is finite. Moreover, for every finite $J \subseteq I$ we consider the finite linear program (I_J) and its dual (II_J):

(I_J)	(II_J)
$\max \sum_{i \in J} \lambda_i c_i$	$\min u^t P$
subject to $\sum_{i \in J} \lambda_i P_i = P$	subject to $u^t P_i \geqq c_i,\ i \in J$
$\lambda_i \geqq 0,\ i \in J$	

Let w be the supremum to (I) where $w = -\infty$ if (I) is inconsistent and z the infimum to (II) where $w = +\infty$ if (II) is inconsistent, and let w_J and z_J be defined analogously from (I_J) and (II_J).

THEOREM 3. (1) *If the constraints of* (II_J) *are consistent for all finite* $J \subseteq I$ *then there is a weak solution* u_k, $k = 1,2,3,\cdots$, *to* (I) *with* $\lim_k u_k^t P = w$;

(2) *For all weak solutions* u_k, $k = 1,2,3,\cdots$, *to* (I) *we have* $\lim_k \inf u_k^t P \geqq w$.

PROOF. (1) Let us assume first that w is finite. Since $w = \sup\{w_J \mid J \subseteq I$ *and* J finite$\} = \sup\{z_J \mid J \subseteq I$ and J finite$\}$, we see that, for every finite set $J \subseteq I$, the set of inequalities

$$u^t(-P) \geqq -w$$
$$u^t P_i \geqq c_i, \quad i \in J$$

has a solution in R^n. By Theorem 1, the semi-infinite system

$$\begin{aligned} u^t(-P) &\geqq -w \\ u^t P_i &\geqq c_i, \quad i \in I \end{aligned}$$

has a solution $u = u(M) \in R(M)^n$. Set $u_k = u(k)$ for all integers k and it is easy to verify that u_k is a weak solution with $\lim_k u_k^t P \leqq w$. Assuming (2) is proved, we have $\lim_k u_k^t P = w$. Also, if $\lambda^{(J)}$ is any (I_J) solution, then $\sum_{i \in J} c_i \lambda_i^{(J)} \leqq \sum_{i \in J} u(M)^t P_i \lambda_i^{(J)} = u(M)^t P \Rightarrow w \leqq u(M)^t P$. Hence $u(M)^t P = w$.

If $w = -\infty$, then $w_J = -\infty$ for all finite $J \subseteq I$, and hence

$$\begin{aligned} u^t(-P) &\geqq +n \text{ for all integers } n = 1, 2, \cdots, |J| \\ u^t P_i &\geqq c_i, \quad i \in J \end{aligned}$$

has a solution in R^n, so that

$$\begin{aligned} u^t(-P) &\geqq +n \text{ for all integers } n \\ u^t P_i &\geqq c_i, \qquad i \in I \end{aligned}$$

has a solution $u = u(M) \in R(M)^n$. Upon setting $u_k = u(k)$ for integers k we see that (since $u(M)^t P$ must have a negative infinite part) $\lim_k u_k^t P = -\infty = w$.

If $w = +\infty$ we use the hypothesis that (II_J) is consistent to insure that (II) has a solution $u = u(M) \in R(M)^n$. Then upon setting $u_k = u(k)$ we see that u_k is a weak solution to (I) and, assuming (2) is proved, that $\lim_k u_k^t P = +\infty = w$.

(2) If $w = -\infty$ there is nothing to prove. Assuming w is finite, for every $\varepsilon > 0$ there is a finite set $J \subseteq I$ such that for suitable λ_i, $i \in J$, we have

$$\begin{aligned} \sum_{i \in J} \lambda_i c_i &\geqq w - \varepsilon \\ \sum_{i \in J} \lambda_i P_i &= P \\ \lambda_i &\geqq 0. \end{aligned}$$

It follows at once that for any weak solution u_k we have

$$\lim_k \inf u_k^t P = \lim_k \inf \left(\sum_{i \in J} \lambda_i u_k^t P_i \right) \geqq$$

$$\sum_{i \in J} \lambda_i \lim_k \inf u_k^t P_i \geqq \sum_{i \in J} \lambda_i c_i \geqq w - \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, the result follows. The case $w = +\infty$, which is similar to the case for w finite, we shall leave for the reader. Q.E.D.

It is known in the case when (I) is consistent and bounded above that there are no duality gaps if one introduces asymptotic solutions to (II), having arbitrarily small errors in any locally convex topology (see Duffin [7], Theorem 1).

Theorem 3* shows that when (I) is consistent and problem (II) is viewed over an ordered field extending R, then a minimum exists for (II) which equals w, including the case $w = +\infty$. Duality gaps are thereby removed without requiring any topology whatever on (II) in a way which substitutes "direct algebraic manipulation and minimal topology"—as promised in [6], p. 784, which, in fact, motivated our work. See also [5].

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* The only case not discussed in Theorem 3 is the case when, for finite $J \subseteq I$, (II_J) is inconsistent. But then $w_J = +\infty$, so $w = +\infty$. Also, (II_J) has no weak solutions, so a fortiori (II) has no weak solutions. Thus, even considering weak solutions, $z = +\infty = w$.